4/27/26

Final Exam
Tuesday, May 5
10:30 - 12:30
in DBRT 209

Similar format to exam 2 (including some proofs). More info next time.

Two additional things about determinants

I. if A is an $n \times n$ matrix then

$$\boxed{A \text{ is invertible iff } \det(A) \neq 0}$$ (important fact)

This is Theorem 6.2.4. It follows from the facts that

(a) elem row operations can change the determinant, but they can't change it from 0 to $\neq 0$ or vice versa

(b) to find A^{-1} from A we started with

$$\left[\, A \mid I_n \,\right] \xrightarrow[\text{operations}]{\text{row}} \left[\, I_n \mid A^{-1} \,\right]$$

Note also $\det(A^{-1}) = \dfrac{1}{\det(A)}$ (related)

Ex For which r does $\begin{bmatrix} 1 & 2 & r \\ 3 & 4 & 5 \\ 6 & 7 & r+r \end{bmatrix}$ fail to be invertible? $\det = -5r + 23 \Rightarrow r = \frac{23}{5}$

II Laplace expansion (aka cofactor expansion)

Describe it by example.

let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 1 & 5 \end{bmatrix}$$

We'll describe how to find $\det(A)$ a different way.

pick any row or column — we'll do 1st row

$$\det A = 1 \cdot \det \begin{bmatrix} 3 & 4 \\ 1 & 5 \end{bmatrix} - 2 \det \begin{bmatrix} 2 & 4 \\ 0 & 5 \end{bmatrix} + 3 \det \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$

$$= 1(15-4) - 2(10-0) + 3(2-0)$$

$$= 11 - 20 + 6 = -3$$

check:

$$\begin{matrix} 1 & 2 & 3 & 1 & 2 \\ 2 & 3 & 4 & 2 & 3 \\ 0 & 1 & 5 & 0 & 1 \end{matrix}$$

$$\det = 15 + 0 + 6 - 0 - 4 - 20$$

$$= 21 - 24 = -3 \quad \checkmark$$

leave to them to check

we could have used a different row or column:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 1 & 5 \end{bmatrix}$$

$$\det A = -2 \det\left(\begin{bmatrix} 2 & 4 \\ 0 & 5 \end{bmatrix}\right) + 3 \det\left(\begin{bmatrix} 1 & 3 \\ 0 & 5 \end{bmatrix}\right) - 1 \det\left(\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\right)$$

$$= -2(10) + 3(5) - 1(-2)$$

$$= -3 \quad \checkmark$$

A few words about eigenvalues and eigenvectors

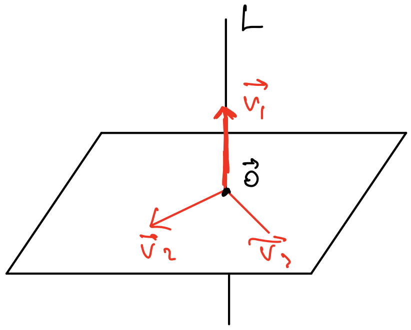
(you're responsible for what we cover)

Recall example from the exam:

if $T: \mathbb{R}^3 \to \mathbb{R}^3$ is defined by reflection about the line L then we can find a basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ of $\mathbb{R}^3$ for which

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(even an orthonormal basis in this case)



$$\Rightarrow \quad [T(\vec{v}_1)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad [T(\vec{v}_2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \qquad [T(\vec{v}_3)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\Rightarrow \quad T(\vec{v}_1) = (1)\,\vec{v}_1 \qquad T(\vec{v}_2) = (-1)\,\vec{v}_2 \qquad T(\vec{v}_3) = (-1)\,\vec{v}_3$$

Def (7.1.1) A lin transf $T: \mathbb{R}^n \to \mathbb{R}^n$ def by an $n \times n$ matrix A is diagonalizable if $\exists$ a basis $\mathcal{B}$ for $\mathbb{R}^n$ so that the matrix $[T]_{\mathcal{B}}$ is diagonal.

> Using change of basis matrices, this means $\exists$ invertible matrix S so that $S^{-1} A S = B$ is diagonal
>
> we'll come back to this to see it's not always possible

Ex/Def $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ $\qquad \mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

$$\Rightarrow \quad T(\vec{v}_1) = 2\,\vec{v}_1 \qquad T(\vec{v}_2) = 3\,\vec{v}_2 \qquad T(\vec{v}_3) = 4\,\vec{v}_3$$

Then 2, 3, 4 are the eigenvalues of T and $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are the eigenvectors of T. $\mathcal{B}$ is called an eigenbasis for T. Caution – the eigenvectors don't always form a basis

Q: How do you find the eigenvalues and the eigenvectors?

Ex $T: \mathbb{R}^3 \to \mathbb{R}^3$ identity. Then $T(\vec{x}) = \vec{x} \quad \forall \vec{x} \in \mathbb{R}^3$

so the only eigenvalue is 1 and any basis is an eigenbasis. So different eigenvectors could corresp. to the same eigenvalue.

So how do we go about finding eigenvalues & eigenvectors?

Say A is an $n \times n$ matrix, and let λ be a scalar.

Then λ is an eigenvalue of A iff $\exists$ nonzero vector $\vec{v} \in \mathbb{R}^n$ s.t.

$$
\begin{aligned}
A\vec{v} = \lambda \vec{v} &\iff A\vec{v} - \lambda \vec{v} = \vec{0} \\
&\iff A\vec{v} - \lambda I_n \vec{v} = \vec{0} \\
&\iff (A - \lambda I)\vec{v} = 0 \\
&\iff \ker(A - \lambda I_n) \neq \{\vec{0}\} \\
&\iff A - \lambda I_n \text{ is not invertible} \\
&\iff \det(A - \lambda I_n) = 0 \qquad (\text{by Th } 6.2.4)
\end{aligned}
$$

Conclusion:

Theorem 7.2.1 Let A be an $n \times n$ matrix. Let $\lambda \in \mathbb{R}$. Then λ is an eigenvalue of A iff $\det(A - \lambda I_n) = 0$.

<u>Def</u> The <u>characteristic equation</u> of A is $\det(A - \lambda I_n) = 0$,

Thinking of λ as a variable (if you prefer, write $\det(A - x I_n)$), and

without "$=0$", it's also called the <u>characteristic polynomial</u> of A.